Spherical Couette Flow of Oldroyd 8-Constant Model

Part II. **Third-order approximation and the stream function** $\Psi^{(3)}$

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Abstract

In the previous work, Part I; [1], the steady flow of an incompressible Oldroyd 8-constant fluid in the annular region between two concentric spheres is investigated up to the second-order approximation. Hence, the normalized second-order velocity field, $\underline{V}$ shows to be

$$\underline{V} = W^{(0)}\hat{\varphi} + \lambda \mathrm{U}_{\perp}^{(1)} + \lambda^{(2)} W^{(2)}\hat{\varphi} + O(\lambda^{3});\ \mathrm{U}_{\perp}^{(1)} = -\nabla \wedge \left(\frac{\Psi^{(1)}}{r \sin\theta} \right) = \hat{r} U^{(1)} + \hat{\theta} V^{(1)}$$

The leading velocity term represents the Newtonian flow in the φ-direction, while the first-order term denoted by the stream–function $\Psi^{(1)}(\mathrm{r},\theta)$, produces a secondary flow field that divides the flow region into four parts symmetric about the z-axis which is the axis of rotation. The second-order approximation gives a viscoelastic contribution $\mathrm{W}^{(2)}(\mathrm{r},\theta)$ in the φ-direction. λ is the retardation time parameter.

The present work is devoted to the solution of the third-order approximation of the same problem treated in [1]. The solution produces a stream–function $\Psi^{(3)}(\mathrm{r},\theta)$ which is being a secondary flow field divides the domain of flow into two similar regions symmetric about an axis perpendicular to the axis of rotation. The streamlines $\Psi^{(3)}(\mathrm{r},\theta) = \mathrm{const.}$ are sketched for Maxwell, Oldroyd-B and Oldroyd 8-constant model fluids; respectively. The results show that the distribution of flow for these fields are mainly affected by the values of their elastic parameters.

Key words: Spherical Couette flow, Oldroyd 8-constant model ,stream–function.

1. Introduction

Spherical Couette flow (SCF) is very interesting research topic in basic hydrodynamics. This flow provides a complex flow for engineering applications. In contrast to cylindrical Couette flow, there is no edge effect or free surface effect on viscoelastic flow characteristics since the fluid is totally enclosed in a gap between two coaxially rotating spheres. Moreover, the SCF is also of interest because the shear rate is varied in the gap space;[1,2], and various flow modes which could not be observed in the case of Newtonian fluids would appear, due to both elastic and inertial hydrodynamic effects of viscoelastic fluids.

For Newtonian fluids the SCF had been studied by many authors, Wimmer [3],Bartels [4], Yavorskaya et al. [5],Nakabayashi [6], Nakabayashi et. al. [7].Recently, SCF has been studied by Nakabayashi [8] .Such flows has been also widely studied for industrial engineering applications and moreover in a basic study of the fluid mechanics, [9].

However, there is very little literature on spherical Couette viscoelastic fluid flow. Recently, in a series of papers Yamaguchi et. al. [9, 10,2] studied this flow.

The present work is devoted to the solution of the third-order approximation of isothermal viscoelastic SCF. This solution, depends on the eight viscometric parameters of the fluid, produces a stream–function $\Psi^{(3)}(r,\theta)$ which is being a secondary flow field divides the flow region between the two spheres into two similar regions. These two regions are being of six stagnation points for Maxwell, Oldroyd-B and Oldroyd 8-constant fluids. This result indicates that the flow field for the viscoelastic fluids is being the same for these three model fluids.

2. Formulation of the problem and the field equations

A viscoelastic fluid moves in the annular region between two concentric spheres of radii R_1 and R_2 $(R_2 > R_1)$. The motion is due to the rotation of the inner sphere with angular velocity Ω about the z-axis, which is located in the center of system, while the outer sphere is kept at rest. The field equations; namely, the constitutive, the continuity and the momentum equations and the boundary conditions are presented as the following.

3. Constitutive, continuity and momentum equations

As presented in [1], we adopted the Oldroyd 8-constant model which relates the stresses and the kinematic variables through the equation; [11, 12, 13].

$$\underline{\underline{t}} + \lambda_1 \overset{\nabla}{\underline{\underline{t}}} + \lambda_3(\underline{\underline{t}}..\underline{\underline{d}} + \underline{\underline{d}}.\underline{\underline{t}}) + \lambda_5(\mathrm{tr}\,\underline{\underline{t}})\underline{\underline{d}} + \lambda_6(\underline{\underline{t}}:\underline{\underline{d}})\underline{\underline{I}} = 2\eta_o\left[\underline{\underline{d}} + \lambda_2\overset{\nabla}{\underline{\underline{d}}} + \lambda_4(\underline{\underline{d}}.\underline{\underline{d}}) + \lambda_7(\underline{\underline{d}}:\underline{\underline{d}})\underline{\underline{I}}\right], \tag{3-1}$$

where $\underline{\underline{t}}$, $\underline{\underline{d}} = \frac{1}{2}[(\underline{\nabla v}) + (\underline{\nabla v})^{\mathrm{T}}]$ and $\underline{\underline{I}}$ are the extra-stress, the rate of deformation and the unit tensors; respectively. The material constants η_o , λ_1 and λ_2 are; respectively, the zero-shear- rate viscosity, the relaxation and retardation times while λ_3 -- λ_7 are further material time constants. The symbol "∇" denotes the upper-convected derivative.

In terms of the spherical polar coordinates $(\tilde{r}, \theta, \varphi)$ the velocity field, independent of the φ-coordinate due to the symmetry about the z-axis, may be written as:

$$\underline{v} = [u(\tilde{r},\theta)\hat{\tilde{r}}, \upsilon(\tilde{r},\theta)\hat{\theta}, w(\tilde{r},\theta)\hat{\varphi}] \quad . \tag{3-2}$$

Consequently, the continuity equation for an incompressible fluid is

$$\tilde{r}^{-2}\partial_{\tilde{r}}\ (\tilde{r}^2 u) + (\tilde{r}\sin\theta)^{-1}\partial_{\theta}(v\sin\theta) = 0, \tag{3-3}$$

allows the definition of a stream-function $\psi(\tilde{r},\theta)$ with the components u and v; i. e.,

$$\underline{u}_{\perp} = u\tilde{r} + v\hat{\theta} = -(\frac{\psi_{,\theta}}{\tilde{r}^2\sin\theta})\tilde{r} + (\frac{\psi_{,\tilde{r}}}{\tilde{r}\sin\theta})\hat{\theta} = -\nabla \wedge (\frac{\psi}{\tilde{r}\sin\theta}\hat{\varphi}), \tag{3-4}$$

which satisfies Eq. (3-3) identically.

We assuming that viscoelasticity of the fluid is dominant such that the inertial term is negligible and hence the momentum equation, for steady state flow, is

$$-\nabla p + \nabla \cdot \underline{\underline{t}} = 0 . \tag{3-5}$$

Substituting from Eq.(3-1) into Eq.(3-5), we obtain

$$-\nabla p + 2\eta_o \nabla \cdot \underline{\underline{d}} - \lambda_2 \begin{bmatrix} \nabla \cdot [\frac{\lambda_1}{\lambda_2}\overset{\nabla}{\underline{\underline{t}}} + \frac{\lambda_3}{\lambda_2}(\underline{\underline{t}}.\underline{\underline{d}} + \underline{\underline{d}}.\underline{\underline{t}}) - 2\frac{\lambda_4}{\lambda_2}\underline{\underline{d}}.\,\underline{\underline{d}} + \frac{\lambda_5}{\lambda_2}(tr\,\underline{\underline{t}})\underline{\underline{d}} - 2\overset{\nabla}{\underline{\underline{d}}}\ ] \\ -\nabla[\frac{\lambda_6}{\lambda_2}(\underline{\underline{t}}:\underline{\underline{d}}) \quad -2\frac{\lambda_7}{\lambda_2}(\underline{\underline{d}}:\underline{\underline{d}})] \end{bmatrix} = 0 \tag{3-6}$$

Introducing the dimensionless variables,

$$\left.\begin{array}{l} r=\tilde{r}/R_1\,;\underline{V}=\underline{v}/\Omega R_1=(U\hat{r},V\hat{\theta},W\hat{\varphi});\ \underline{\underline{D}}=\underline{\underline{d}}/\Omega\ ;\underline{\underline{T}}=\underline{\underline{t}}/\eta_0\Omega;P=p/\eta_0\Omega \\ \Psi=\dfrac{\psi}{R_1^3\Omega};\quad \lambda=\lambda_2\Omega;\xi_i=\dfrac{\lambda_i}{\lambda_2}\quad for\ \ i=1,3,4,5,6,7 \end{array}\right\} \qquad (3\text{-}7)$$

into Eqs. (3-1), (3-4) and (3-6) and making use of some manipulation ; [1], we get

$$\underline{\underline{T}}=2\underline{\underline{D}}-\lambda\left[\xi_1\overset{\nabla}{\underline{\underline{T}}}+\xi_3(\underline{\underline{T}}.\underline{\underline{D}}+\underline{\underline{D}}.\underline{\underline{T}})-2\xi_4\underline{\underline{D}}.\underline{\underline{D}}+\xi_5(tr\,\underline{\underline{T}})\underline{\underline{D}}-2\overset{\nabla}{\underline{\underline{D}}}+(\xi_6\underline{\underline{T}}:\underline{\underline{D}}-2\xi_7\underline{\underline{D}}:\underline{\underline{D}})\underline{\underline{I}}\right], \quad (3\text{-}8a)$$

$$\frac{1}{r^2}\left[\partial_r(r^2W_{,r})+\partial_\theta[\frac{1}{\sin\theta}\partial_\theta(W\sin\theta)]\right]-\lambda\Lambda_3=0, \qquad (3.8b)$$

$$\left[\partial_r^2+\frac{\sin\theta}{r^2}\partial_\theta[\frac{1}{\sin\theta}\partial_\theta]\right]^2\Psi-\lambda\sin\theta[\partial_r(r\Lambda_2)-\partial_\theta\Lambda_1]=0, \qquad (3\text{-}8c)$$

where, the non-dimensionality is obtained by using R_1, Ω and $\eta_0\Omega$ as the characteristic length, time and stress; respectively, and $\underline{\underline{T}}_{exc}=-P\underline{\underline{I}}+\underline{\underline{T}}$, is the extra-stress .

The perturbation term, $\{\ \}$, is abbreviated in Eq. (3-6) by the dimensionless vector $\underline{\Lambda}(r,\theta)$ and the scalar $G(r,\theta)$ defined by the expressions

$$\underline{\Lambda}(r,\theta)=\nabla\cdot\left[\xi_1\overset{\nabla}{\underline{\underline{T}}}+\xi_3(\underline{\underline{T}}.\underline{\underline{D}}+\underline{\underline{D}}.\underline{\underline{T}})-2\xi_4(\underline{\underline{D}}.\underline{\underline{D}})+\xi_5(tr\,\underline{\underline{T}})\underline{\underline{D}}-2\overset{\nabla}{\underline{\underline{D}}}\right],\ G(r,\theta)=\left\lfloor\xi_6\underline{\underline{T}}:\underline{\underline{D}}-2\xi_7\underline{\underline{D}}:\underline{\underline{D}}\right\rfloor$$

(3-8d)

where

$\underline{\Lambda}_\perp=\Lambda_1\hat{r}+\Lambda_2\hat{\theta}$ and $\Lambda_3=\hat{\varphi}\cdot\underline{\Lambda}$.

The boundary conditions imposed on the functions W and Ψ are

$W=\sin\theta\ \ ,0$ at $r=1,\ a$; $a=R_2/R_1$, (3-9a)

$\Psi=\Psi_{,r}=0,0$ at $r=1,a$. (3-9b)

The functions W and Ψ , Eqs. (3-8b) and (3-8c), must be determined such that the boundary conditions (3-9a) and (3-9b) are satisfied. Using the method of solution which adopted in [1], see on the Appendix , the final form of the third-order solution is given as the flowing.

4.Third-order approximation

As indicated in [1], the same procedure of the solution will be adapted. Taking $n=3$, the coefficient of λ^3 in Eqs. (A-3) and (A-4) we get:

$$\frac{1}{r^2}\partial_r(r^2W^{(3)}_{,r})+\partial_\theta\frac{1}{\sin\theta}\partial_\theta(W^{(3)}\sin\theta)-(\Lambda_3^{(2,0)}+\Lambda_3^{(1,1)}+\Lambda_3^{(0,2)})=0, \qquad (4\text{-}1)$$

$$\left[\partial_r^2+\frac{\sin\theta}{r^2}\partial_\theta[\frac{1}{\sin\theta}\partial_\theta]\right]^2\Psi^{(3)}-\sin\theta\left[\partial_r[r(\Lambda_3^{(2,0)}+\Lambda_3^{(1,1)}+\Lambda_3^{(0,2)})]-\partial_\theta(\Lambda_2^{(2,0)}+\Lambda_2^{(1,1)}+\Lambda_2^{(0,2)})\right]=\text{zero} \tag{4-2}$$

Hence,

$$\begin{aligned}\Lambda_3^{(2,0)} &= r^{-3}\partial_r[r^3(\zeta_1\overset{\nabla}{T}{}^{(2,0)}_{r\varphi}-2\overset{\nabla}{T}{}^{(2,0)}_{r\varphi}+\xi_3(\overset{(2)}{T}_{rr}\overset{(0)}{D}_{r\varphi}+\overset{(2)}{T}_{r\theta}\overset{(0)}{D}_{\theta\varphi}+\overset{(2)}{T}_{r\varphi}\overset{(0)}{D}_{\varphi\varphi}+\overset{(2)}{D}_{rr}\overset{(0)}{T}_{r\varphi}+\overset{(2)}{D}_{r\theta}\overset{(0)}{T}_{\theta\varphi}+\overset{(2)}{D}_{r\varphi}\overset{(0)}{T}_{\varphi\varphi})\\
&\quad -2\xi_4(\overset{(2)}{D}_{rr}\overset{(0)}{D}_{r\varphi}+\overset{(2)}{D}_{r\theta}\overset{(0)}{D}_{\theta\varphi}+\overset{(2)}{D}_{r\varphi}\overset{(0)}{D}_{\varphi\varphi})+\zeta_5\Gamma^{(2)}D^{(0)}_{r\varphi})]+\frac{1}{r\sin^2\theta}\partial_\theta[\sin^2\theta(\zeta_1\overset{\nabla}{T}{}^{(2,0)}_{r\varphi}-2\overset{\nabla}{D}{}^{(2,0)}_{\theta\varphi}\\
&\quad +\xi_3(\overset{(2)}{T}_{\varphi r}\overset{(0)}{D}_{r\theta}+\overset{(2)}{T}_{\varphi\theta}\overset{(0)}{D}_{\theta\theta}+\overset{(2)}{T}_{\varphi\varphi}\overset{(0)}{D}_{\theta\varphi}+\overset{(2)}{D}_{\varphi r}\overset{(0)}{T}_{r\theta}+\overset{(2)}{D}_{\varphi\theta}\overset{(0)}{T}_{\theta\theta}+\overset{(2)}{D}_{\varphi\varphi}\overset{(0)}{T}_{\varphi\theta})\\
&\quad -4\xi_4(\overset{(2)}{D}_{\varphi r}\overset{(0)}{D}_{r\theta}+\overset{(2)}{D}_{\varphi\theta}\overset{(0)}{D}_{\theta\theta}+\overset{(2)}{D}_{\varphi\varphi}\overset{(0)}{D}_{\theta\varphi})+\zeta_5\Gamma^{(2)}\overset{(0)}{D}_{\theta\varphi})]\end{aligned} \tag{4-3a}$$

$$\begin{aligned}\Lambda_3^{(1,1)} &= r^{-3}\partial_r[r^3(\zeta_1\overset{\nabla}{T}{}^{(1,1)}_{r\varphi}-2\overset{\nabla}{D}{}^{(1,1)}_{r\varphi}+\xi_3(\overset{(1)}{T}_{rr}\overset{(1)}{D}_{r\varphi}+\overset{(1)}{T}_{r\theta}\overset{(1)}{D}_{\theta\varphi}+\overset{(1)}{T}_{r\varphi}\overset{(1)}{D}_{\varphi\varphi}+\overset{(1)}{D}_{rr}\overset{(1)}{T}_{r\varphi}+\overset{(1)}{D}_{r\theta}\overset{(1)}{T}_{\theta\varphi}+\overset{(1)}{D}_{r\varphi}\overset{(1)}{T}_{\varphi\varphi})\\
&\quad -2\xi_4(\overset{(1)}{D}_{rr}\overset{(1)}{D}_{r\varphi}+\overset{(1)}{D}_{r\theta}\overset{(1)}{D}_{\theta\varphi}+\overset{(1)}{D}_{r\varphi}\overset{(1)}{D}_{\varphi\varphi})+\zeta_5\Gamma^{(1)}D^{(1)}_{r\varphi})]+\frac{1}{r\sin^2\theta}\partial_\theta[\sin^2\theta(\zeta_1\overset{\nabla}{T}{}^{(1,1)}_{r\varphi}-2\overset{\nabla}{D}{}^{(1,1)}_{\theta\varphi}\\
&\quad +\xi_3(\overset{(1)}{T}_{\varphi r}\overset{(1)}{D}_{r\theta}+\overset{(1)}{T}_{\varphi\theta}\overset{(1)}{D}_{\theta\theta}+\overset{(1)}{T}_{\varphi\varphi}\overset{(1)}{D}_{\theta\varphi}+\overset{(1)}{D}_{\varphi r}\overset{(1)}{T}_{r\theta}+\overset{(1)}{D}_{\varphi\theta}\overset{(1)}{T}_{\theta\theta}+\overset{(1)}{D}_{\varphi\varphi}\overset{(1)}{T}_{\varphi\theta})\\
&\quad -4\xi_4(\overset{(1)}{D}_{\varphi r}\overset{(1)}{D}_{r\theta}+\overset{(1)}{D}_{\varphi\theta}\overset{(1)}{D}_{\theta\theta}+\overset{(1)}{D}_{\varphi\varphi}\overset{(1)}{D}_{\theta\varphi})\ +\zeta_5\Gamma^{(1)}D^{(1)}_{\theta\varphi})]\end{aligned} \tag{4-3b}$$

$$\begin{aligned}\Lambda_3^{(0,2)} &= r^{-3}\partial_r[r^3(\zeta_1\overset{\nabla}{T}{}^{(0,2)}_{r\varphi}-2\overset{\nabla}{D}{}^{(0,2)}_{r\varphi}+\xi_3(\overset{(0)}{T}_{rr}\overset{(2)}{D}_{r\varphi}+\overset{(0)}{T}_{r\theta}\overset{(2)}{D}_{\theta\varphi}+\overset{(0)}{T}_{r\varphi}\overset{(2)}{D}_{\varphi\varphi}+\overset{(0)}{D}_{rr}\overset{(2)}{T}_{r\varphi}+\overset{(0)}{D}_{r\theta}\overset{(2)}{T}_{\theta\varphi}+\overset{(0)}{D}_{r\varphi}\overset{(2)}{T}_{\varphi\varphi})\\
&\quad -2\xi_4(\overset{(0)}{D}_{rr}\overset{(2)}{D}_{r\varphi}+\overset{(0)}{D}_{r\theta}\overset{(2)}{D}_{\theta\varphi}+\overset{(0)}{D}_{r\varphi}\overset{(2)}{D}_{\varphi\varphi})+\zeta_5\Gamma^{(0)}D^{(2)}_{r\varphi})]+\frac{1}{r\sin^2\theta}\partial_\theta[\sin^2\theta(\zeta_1\overset{\nabla}{T}{}^{(0,2)}_{\theta\varphi}-2\overset{\nabla}{D}{}^{(0,2)}_{\theta\varphi}\\
&\quad +\xi_3(\overset{(0)}{T}_{\varphi r}\overset{(2)}{D}_{r\theta}+\overset{(0)}{T}_{\varphi\theta}\overset{(2)}{D}_{\theta\theta}+\overset{(0)}{T}_{\varphi\varphi}\overset{(2)}{D}_{\theta\varphi}+\overset{(0)}{D}_{\varphi r}\overset{(2)}{T}_{r\theta}+\overset{(0)}{D}_{\varphi\theta}\overset{(2)}{T}_{\theta\theta}+\overset{(0)}{D}_{\varphi\varphi}\overset{(2)}{T}_{\varphi\theta})\\
&\quad -4\xi_4(\overset{(0)}{D}_{\varphi r}\overset{(2)}{D}_{r\theta}+\overset{(0)}{D}_{\varphi\theta}\overset{(2)}{D}_{\theta\theta}+\overset{(0)}{D}_{\varphi\varphi}\overset{(2)}{D}_{\theta\varphi})+\zeta_5\Gamma^{(0)}D^{(2)}_{\theta\varphi})]\end{aligned} \tag{4-3c}$$

which reduces to:

$$\Lambda_3^{(2,0)}=\Lambda_3^{(1,1)}=\Lambda_3^{(0,2)}=\text{zero}. \tag{4-3d}$$

Hence ,

$$\frac{1}{r^2}[\partial_r(r^2W^{(3)}_{,r})+\partial_\theta\frac{1}{\sin\theta}\partial_\theta(W^{(3)}\sin\theta)]=0 \tag{4-4a}$$

with the boundary conditions

$$W^{(3)}=\begin{Bmatrix}0\\0\end{Bmatrix}\quad \text{at } r=\begin{Bmatrix}1\\a\end{Bmatrix} \tag{4-4b}$$

The solution of (4-4a) with the boundary conditions (4-4b) is being

$$W^{(3)}=\text{zero} \tag{4-5}$$

In the same manners, for the stream-function $\Psi^{(3)}(r,\theta)$ we get:

$$\begin{aligned}\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right]^2\Psi^{(3)} = L^3\cos\theta\{16[(3r^{-4}+(-4+7\beta')r^{-5}-(32+40\beta')r^{-7}+(406-285\beta')r^{-9}\\ -(550+32\beta')r^{-10}+(67-360\beta')r^{-12}+(102+1454\beta')r^{-13}]\cos^4\theta\\ +[\ ((-16+14\beta')r^{-2}-68r^{-4}+(130-206\beta')r^{-5}+(15652+2496\beta')r^{-7})\\ +(-8300+12152\beta')r^{-9}+(10792-4\beta')r^{-10}\\ +(-10844+17360\beta')r^{-12}-(59552+52732\beta')r^{-13}]\cos^2\theta\\ +[\ (-4-8\beta')r^{-2}+20r^{-4}+(-58+86\beta')r^{-5}-(15139+1856\beta')r^{-7}\\ +(2485-7593\beta')r^{-9}+(-1992+517\beta')r^{-10}\\ +(9772-11600\beta')r^{-12}+(57921+29468\beta')r^{-13}]\},\end{aligned}$$

(4-6a)

$$L=[(\xi_1-1)-(\xi_3-\xi_4)] \qquad \text{(4-6b)}$$

$$\beta=[(\xi_7-\xi_6)(\xi_1-\xi_3-\tfrac{3}{2}\xi_5)-(\xi_1-1)(\xi_3+\xi_5)+(\xi_3-\xi_4)(\xi_1-\xi_3-\xi_5)] \text{ and } \beta'=\frac{\beta}{L^2}\ . \qquad \text{(4-6c)}$$

with the boundary conditions,

$$\Psi^{(3)}=\Psi^{(3)}_{,r}=\begin{Bmatrix}0\\0\end{Bmatrix} \text{ at } r=\begin{Bmatrix}1\\a\end{Bmatrix}, \qquad \text{(4-6d)}$$

The solution of Eq. (4-6a) for the stream-function $\Psi^{(3)}(r,\theta)$ subject to the boundary conditions (4-6d) with a = 2 for simplicity; is given by:

$$\begin{aligned}\Psi^{(3)} = L^3\cos\theta\{16[(0.00006-0.0001\beta')r^7+(-0.00036+0.0009\beta')r^5+0.0107+(-0.0277+0.0486\beta')r^{-1}-\\ (0.0202+0.4402\beta')r^{-2}+(0.4-0.5\beta')r^{-3}+(-1.064+0.6019\beta')r^{-4}+(1.1555-0.7916\beta')r^{-5}-\\ (0.4807+0.0279\beta')r^{-6}+(0.0143-0.0769\beta')r^{-8}+(0.013+0.1854\beta')r^{-9}]\cos^4\theta+\\ [(-0.0013+0.0033\beta')r^7+(-0.4435+1.1007\beta')r^5+(3.0343-8.7564\beta')r^3+(-0.6666+0.5833\beta')r^2+\\ (-0.1246+49.5817\beta')+(0.972-16.3611\beta')r^{-1}-(75.1539+36.5626\beta')r^{-2}+(94.472-0.4444\beta')r^{-3}+\\ (0.4637-13.759)r^{-4}+(-25.4055+22.7933\beta')r^{-5}+(10.24+0.32987\beta')r^{-6}+\\ (-1.6839+3.0885\beta')r^{-8}-(5.7037+1.5971\beta')r^{-9}]\cos^2\theta+\\ [(0.0004-0.309\beta')r^7+(0.4487-1.1153\beta')r^5+(-34.0709+40.9796\beta')r^3+(-296.883+294.669\beta')r+\\ (167.47-169.859\beta')r^2+(176.501-228.026\beta')+(-2.1388+32.4167\beta')r^{-1}+(75.4762+43.6064\beta')r^{-2}-\\ (100.869-7.5555\beta')r^{-3}+(2.2465+4.1277\beta')r^{-4}+(7.417-10.1273\beta')r^{-5}+(-2.5477+0.1180\beta')r^{-6}+\\ (1.4549-1.8578\beta')r^{-8}+(5.4956+2.6247\beta')r^{-9}]\}\end{aligned}$$

(4-7)

The streamlines of $\Psi^{(3)}(r,\theta)=\text{const.}$ is being a secondary flow divides the annular spherical gap into two similar flow regions symmetric about an axis perpendicular to the z-axis. This flow includes all the material parameters in the employed Oldroyd 8-constant constitutive equation; namely $\eta_0, \lambda_i; \quad i = 1, 2, \ldots\ldots, 7$ and hence it is a viscoelastic flow.

Inspection of Eq. (4-7) show –from a mathematical point of view - that the stream-function $\Psi^{(3)}(r,\theta)$ is consists of two parts, the first one is proportional to $[(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4)]^3$,which is purely positive factor since $\lambda_1 > \lambda_2$, and the second term $(\lambda_1 - \lambda_2)[(\lambda_1 - \lambda_2).(\lambda_3 + \lambda_5) + (\lambda_3 - \lambda_4)(\lambda_1 - \lambda_3 - \lambda_5) - (\lambda_7 - \lambda_6)(\lambda_1 - \lambda_3 - \frac{3}{2}\lambda_5)]$ is purely negative . Hence, we notice that for $\beta = 0$; i.e. in the presence of η_0, λ_1(Maxwell fluid) or η_0, $\lambda_1$and λ_2 (Oldroyd-B fluid) the flow is enhanced in the two cases while it is diminished for $\beta \neq 0$; i.e. for more complex flow.

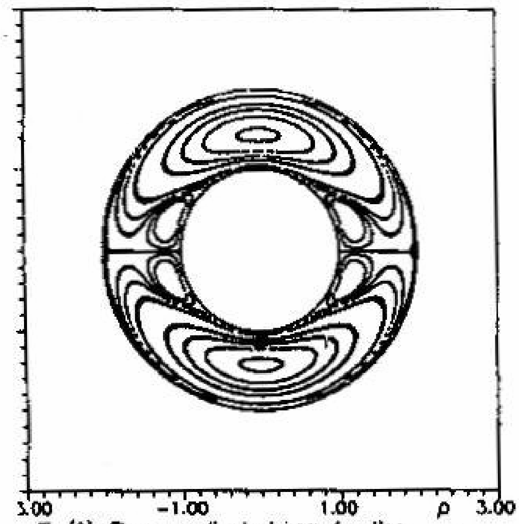

Fig.(1): The normalized stream function $\psi^{(2)}(r,\vartheta)$for Maxwell fluid

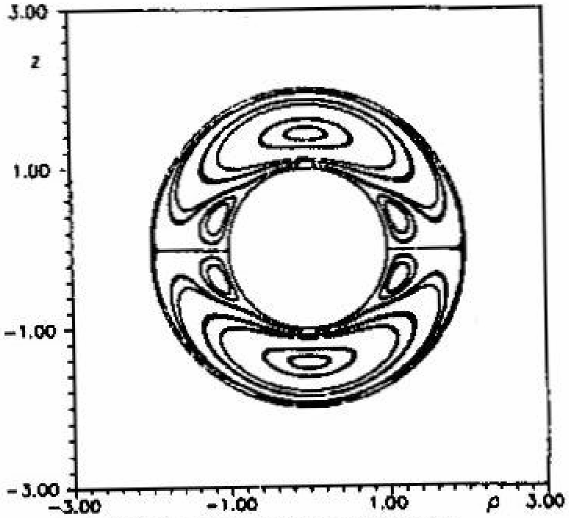

Fig.(2): The normalized stream function $\psi^{(2)}(r,\vartheta)$for Oldroyd-B fluid $\lambda_1-\lambda_2=0.5$

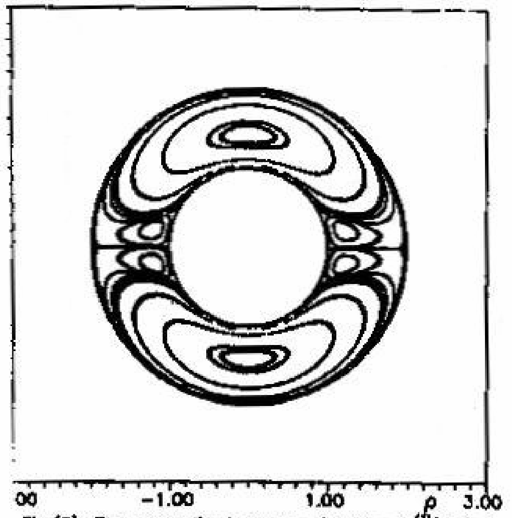

Fig.(3): The normalized stream function $\psi^{(2)}(r,\vartheta)$: for Oldroyd B-const. fluid $\beta=-0.5$ $\lambda_1-\lambda_2=0.5$

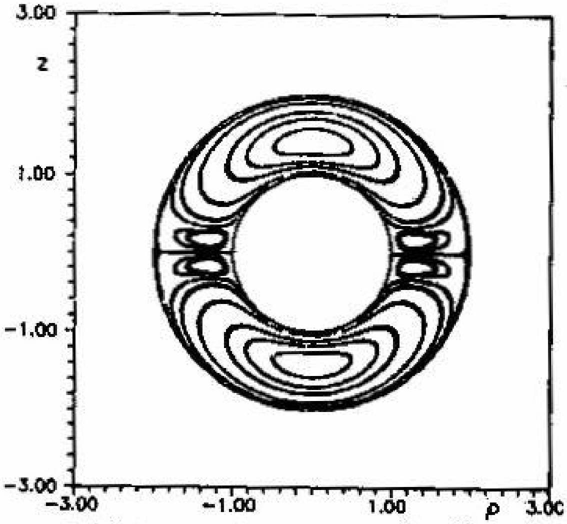

Fig.(4): The normalized stream function $\psi^{(2)}(r,\vartheta)$ for Oldroyd B-const. fluid $\beta=-1$ $\lambda_1-\lambda_2=0.5$

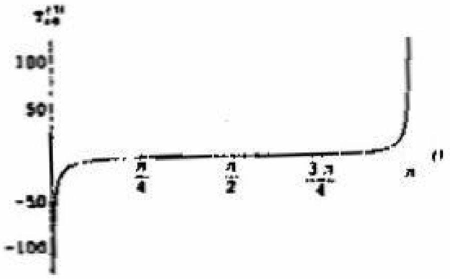

Fig. (5 a) : The stress component $T^{(3)}_{r\theta}$ versus θ
r = 1.1, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

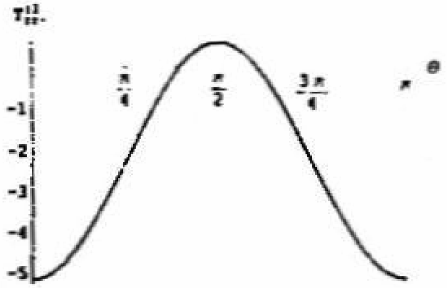

Fig. (5 b) : The stress component $T^{(3)}_{rr}$ versus θ ,
r = 1.1, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

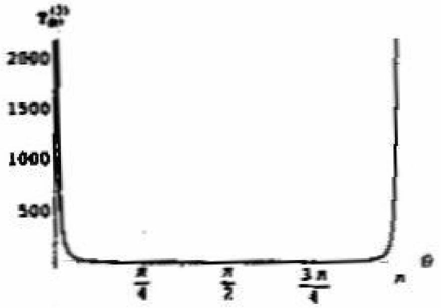

Fig. (5 c) : The stress component $T^{(3)}_{\theta\theta}$ versus θ ,
r = 1.1, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

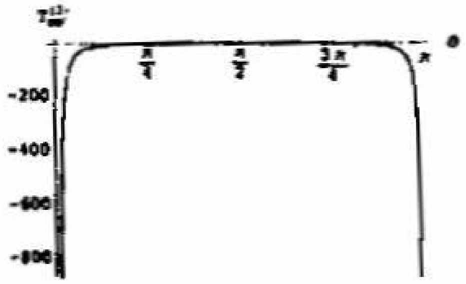

Fig. (5 d) : The stress component $T^{(3)}_{\varphi\varphi}$ versus θ ,
r = 1.1, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

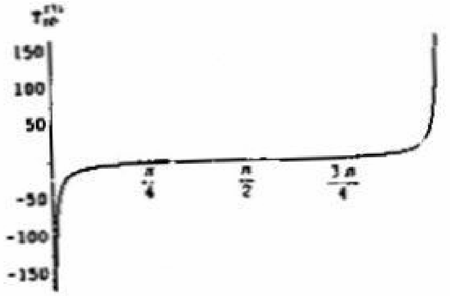

Fig. (5 e) : The stress component $T^{(3)}_{r\theta}$ versus θ ,
r = 2, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

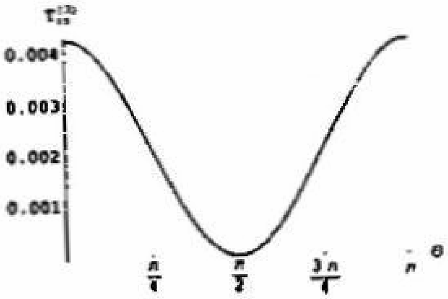

Fig. (5 f) : The stress component $T^{(3)}_{rr}$ versus θ ,
r = 2, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

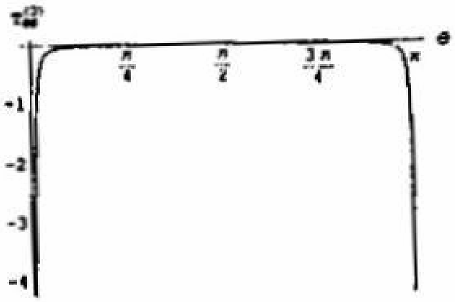

Fig. (5 g) : The stress component $T^{(3)}_{\theta\theta}$ versus θ ,
r = 2, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

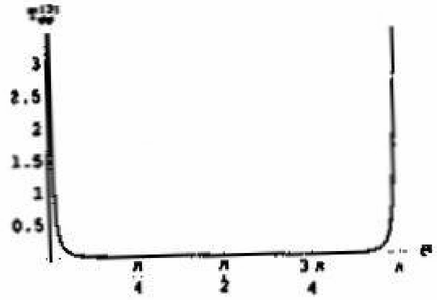

Fig. (5 h) : The stress component $T^{(3)}_{\varphi\varphi}$ versus θ
r = 2, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = 0$.

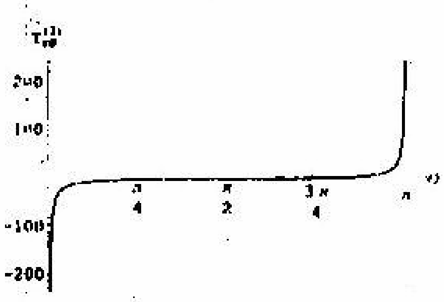

Fig. (6 a) : The stress component $T^{(3)}_{r\theta}$ versus θ , $r = 1.1$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

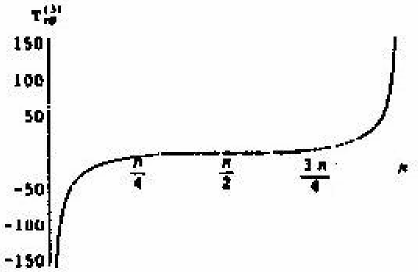

Fig. (6 e) : The stress component $T^{(3)}_{r\theta}$ versus θ , $r = 2$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

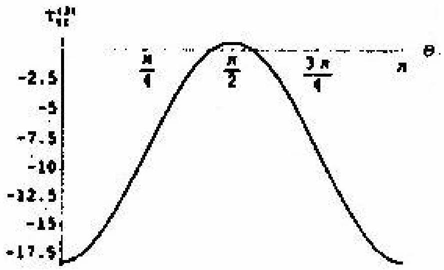

Fig. (6 b) : The stress component $T^{(3)}_{zz}$ versus θ , $r = 1.1$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

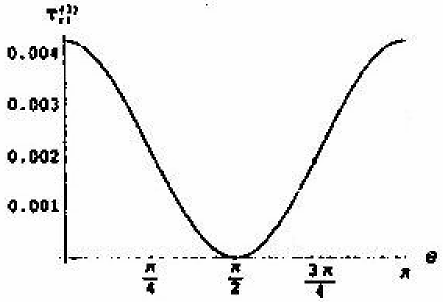

Fig. (6 f) : The stress component $T^{(3)}_{zz}$ versus θ , $r = 2$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

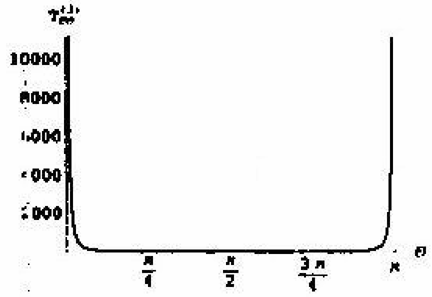

Fig. (6 c) : The stress component $T^{(3)}_{\theta\theta}$ versus θ , $r = 1.1$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

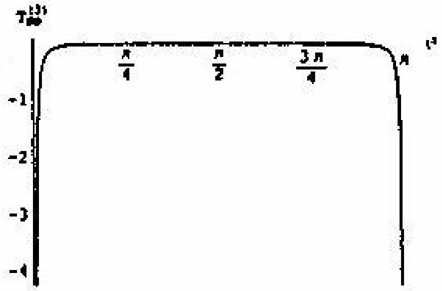

Fig. (6 g) : The stress component $T^{(3)}_{\theta\theta}$ versus θ , $r = 2$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

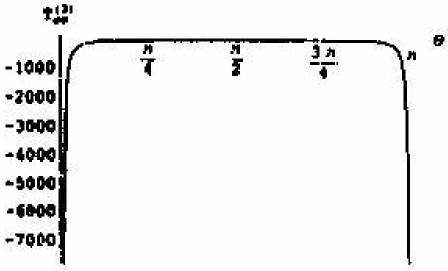

Fig. (6 d) : The stress component $T^{(3)}_{\phi\phi}$ versus θ , $r = 1.1$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

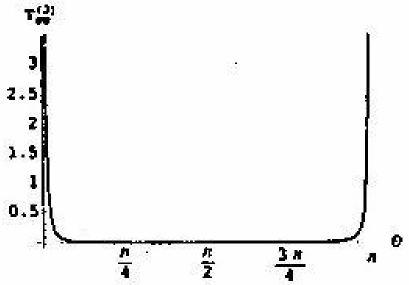

Fig. (6 h) : The stress component $T^{(3)}_{\phi\phi}$ versus θ , $r = 2$, $(\lambda_1 - \lambda_2) = 0.5$ and $\beta = -1$.

Conclusion

The behavior of a viscoelastic fluid in the SCF where the inner sphere is rotating with angular velocity Ω and the outer sphere is kept at rest has been investigated in Part I; [1]. The solution is performed up to the second-order through the expansion of the dynamical variables in power series of the dimensionless retardation time . For continuity, the present work is devoted to the solution of the third-order approximation. The mapping of this field in the $\rho z-$plane is being a secondary flow which divides the annular region of the SCF into two similar regions symmetric about an axis perpendicular to the z-axis; i.e. the ρ -axis.

The results show that the stream function $\Psi^{(3)}$ is a viscoelastic term depends on all of the material parameters; i.e. $\eta_0, \lambda_i, i=1,2,........,7$. Figure (1) represents the streamlines $\Psi^{(3)} = \text{const.}$ for Maxwell fluid. The domain of flow between the two spheres is divided into two similar regions with two large loops and four small eddies having six stagnation points. The same behavior for Oldroyd-B fluid; Fig. (2), Oldroyd 8-constant fluid , $\beta = -0.5$; Fig. (3), and finally for Oldroyd 8-constant fluid , $\beta = -1$; Fig. (4). This means that for higher order perturbation, the flow field for viscoelastic fluids is being the same for Maxwell, Oldroyd-B and Oldroyd 8-constant model fluids. As it is expected, the torque field is unaffected by the third-order perturbation.

Finally, the non-vanishing third-order stresses at the surface of the stationary outer sphere are being $T^{(3)}_{rr}$, $T^{(3)}_{\theta\theta}$, $T^{(3)}_{\varphi\varphi}$ and $T^{(3)}_{r\theta}$. These components are calculated and then plotted in Figs.(5a,..5h,6a,..6h) . In view of these figures we notice that all these components are symmetric about $\theta = \pi/2$ and the behavior for fluids with $\beta = 0$; i.e. Oldroyd-B fluids is the same as that for higher order elastic fluids; say $\beta = -1$in the present work, except in their values only . An inversion of the normal stress- components directions for r=1.1 and r=2 ; respectively is shown.

Appendix

Method of successive approximation

According to the method of perturbation employed in [1], the functions $W, \Psi, \underline{\underline{D}}, \underline{\underline{T}}$ or P are expanded in power series of the retardation parameter λ as follows

$$A = \sum_{n=0} \lambda^n A^{(n)} \qquad \text{(A-1)}$$

where A may represents any of the above functions. Consequently, the expansion of Eqs.(3-8a),(3-8b) and (3-8c) are written as:

$$\sum_{n=0} \lambda^n \Bigg[\underline{\underline{T}}^{(n)} - 2\underline{\underline{D}}^{(n)} + \lambda \sum_{k \le n} [\xi_1 \overset{\nabla}{\underline{\underline{T}}}{}^{(n-k,k)} - 2 \overset{\nabla}{\underline{\underline{D}}}{}^{(n-k,k)} + \xi_3 (\underline{\underline{T}}^{(n-k)} . \underline{\underline{D}}^{(k)} + \underline{\underline{D}}^{(k)} . \underline{\underline{T}}^{(n-k)}) - \xi_4 \underline{\underline{D}}^{(n-k)} . \underline{\underline{D}}^{(k)} +$$

$$\xi_5 (tr \underline{\underline{T}}^{(n-k)}) \underline{\underline{D}}^{(k)} + (\xi_6 \underline{\underline{T}}^{(n-k)} : \underline{\underline{D}}^{(k)} - 2\xi_7 \underline{\underline{D}}^{(n-k)} : \underline{\underline{D}}^{(k)}) \underline{\underline{I}}] \Bigg] = 0$$

(A-2)

$$\sum_{n=0} \lambda^n \left[\frac{1}{r^2} \partial_r (r^2 W^{(n)}_{,r}) + \partial_\theta [\frac{1}{\sin\theta} \partial_\theta (W^{(n)} \sin\theta)] - \lambda \sum_{k \le n} \Lambda_3^{(n-k,k)} \right] = 0, \qquad \text{(A-3)}$$

and

$$\sum_{n=0} \lambda^n \left[[\partial_r^2 + \frac{\sin\theta}{r^2} \partial_\theta (\frac{1}{\sin\theta} \partial_\theta)]^2 \Psi - \lambda \sin\theta \sum_{k \le n} [\partial_r (r \Lambda_2^{(n-k,k)}) - \partial_\theta \Lambda_1^{(n-k,k)}] \right] = 0, \qquad \text{(A-4)}$$

The required boundary conditions (3-9a) and (3-9b) are reduced to:

$$W^{(k)} = \delta_{0n} \sin\theta, 0 \quad \text{for} \quad r = 1 \ , \ a \ , \qquad \text{(A-5a)}$$

$$\Psi^{(k)} = \Psi^{(k)}_{,r} = 0,0 \quad \text{for} \quad r = 1 \ , \ a \ . \qquad \text{(A-5b)}$$

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